AN

# ESSAY ON THE ELEMENTS

OF

# CYCLOMETRY,

INTENDED TO ILLUSTRATE THE RELATIONS WHICH EXIST BETWEEN

## CURVES AND RIGHT-LINED FIGURES.

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BY JOHN LUCCOCK,

AUTHOR OF "A TREATISE ON BRITISH WOOL," "LETTER ON THE SPANISH REVOLUTION," "NOTES ON RIO DE JANEIRO AND THE SOUTHERN PARTS OF BRAZIL," &c. &c.


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# DEDICATION

TO THE

## PRESIDENT, MANAGERS, AND MEMBERS

OF THE

## Leeds Mechanics' Institution.

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GENTLEMEN,

PRESUMING that an Essay professedly elementary may be dedicated not improperly, nor unacceptably, to those who have associated for the very important purpose of promoting a knowledge of scientific principles, or with the still more laudable design of making themselves acquainted with the application of such principles to the common concerns of life, I present myself (a known and plain mechanic) to lay before you a small pamphlet, with a sincere wish that it may promote your views, by stimulating some to emulation and others to afford the means of improvement.

Nor is it possible, I apprehend, to ask for more powerful patronage than that of a Society, in which no interests clash, no designs can prevail but those which tend to mutual advantage, the only solid basis of social-order and happiness. You are experimentally convinced that the interest of work-people and their employers is one and inseparable, that they must both rise in the scale of society together, or, that by counteracting each other, both must suffer.

You still have a very few objectors to your plans, a still smaller number of opposers; but such persons, I conceive, must be ignorant of the great lessons taught by history, respecting manufacturing districts in every country of Europe, and in every period of modern time. They must be inobservant of the quality of our additional population, of the ignorant and vicious parts of the empire whence it is collected; they must have forgotten, what I am sure you still remember, that in periods of public pressure, the peace and good order of this neighbourhood was mainly preserved by the good sense of the suffering work-people themselves, when disorganization was triumphant around us. They forget likewise that times of pressure may, and must, return again, (may they be late,) and that a greater mass of good sense, the effect of education, will then be required to keep in check the discontented, the turbulent, and the designing.

Convinced that this is the natural and undeviating tendency of mental culture, I give to such Societies as yours the whole of my feeble support, have enrolled my name upon your lists, and am desirous of being esteemed,

Gentlemen,

Your very humble and most devoted servant,

THE AUTHOR.

*Leeds, July 21st,* 1825.

# PREFACE.

About eighteen months ago the pith of the following pages, thrown as a trifle of some value amidst a heap of rubbish, was printed with an Introduction, stating "that, "during severe and protracted illness, the writer attempted "to alleviate the tedium of confinement by Mathematical "Recollections; that he condensed into a small compass "and printed them, because, to his untaught and perhaps "debilitated mind, they seemed to contain something new "and useful." He was soon told however that the argument in Prop. III. was inconclusive; that the doctrine itself was not true, and that the error might be rendered manifest by several different methods. He was aware indeed that the Proposition contradicted some generally received opinions, but thinking that possibly it might do so without being very erroneous, he re-examined, with all the diligence he was master of, not only the proposition itself, but also, every step of a very long argument, which led to that conclusion, and must even now confess himself unable to perceive either that he has drawn a false deduction, on some particular point, or that, upon the whole, he has merely argued in a Circle.

For this reason, with all due deference to better opinions than his own, he now lays a short abstract of the entire argument before the public, desirous that the Essay may meet with only such attention as the subject and his mode of treating it may be found to merit. One of the Propositions, the groundwork of some others, is very ancient, but the author became acquainted with it from modern sources, and is not at all aware that any other person has made use of it, in the way he has done, (he hopes with some success,) to ascertain the relations which subsist between the Circle and Right-lined Figures. In the very limited course of his mathematical reading nothing of the kind has fallen under his own notice.

Yet it does not become him, in such a case, to be confident. He may have lost his clue, and be still wandering amidst the hopeless mazes of a labyrinth, where multitudes of the most observant have become irretrievably bewildered. To fail, however, where the wisest have erred is no disgrace; and, in the smallest degree, to succeed offers a temptation which it is presumed few well-formed minds are desirous, or able, to resist.

The author thinks it right, however, to suggest, that in his opinion, nothing ought to be admitted as conclusive against his argument, which does not amount to *positive* and *geometrical* proof. So soon as this is adduced, he will gladly lay aside his present opinion, and acknowledge that he has spent his time in an idle pursuit. It requires all the nerve he possesses to persist in maintaining a doctrine, generally deemed incorrect and fanciful, to expose himself to the contempt almost universally poured by the learned, and especially by the ignorant, upon persons who like him, venture out of their sphere, to appear, a second time

(conscious that his first attempt has been discouraged,) as the advocate of a question undoubtedly encumbered with difficulties, which for many ages have baffled the best efforts of the greatest minds, and aware that the weight of these difficulties can be estimated, and even rudely guessed at, only by those who have many times attempted to remove them.

Perhaps the want of arrangement in this Essay may be justly complained of; in fact, the propositions are only the most important links of a long chain of argument, in which the author followed truth wherever he thought her visible, though dimly seen, and however devious and intricate the path might be through which she conducted him. In the first edition he had accidentally written in the form of Proposition, Construction and Demonstration, and, in order to give a sort of conformity to the whole, has persevered in the same plan, endeavouring as he advanced, to render the dependence of each deduction, upon some truth previously ascertained, as obvious as the nature of such an abstract would permit. He has frequently thrown into the form of Corollary and Notes what originally stood as independent and detached parts of the subject. Indeed several of the deductions were in the first instance Algebraic, and a good proportion of the labour required in condensation, has been spent in contriving the means of expressing them in a Geometrical form.

The Diagrams may appear somewhat more crowded than is generally deemed desirable, yet it was proper to have as few of them as possible, consistent with distinctness, and the pains taken to describe minutely every part of the Construction, it is hoped, will prevent, in a good measure at least, any obscurity which might arise from the circum-

stance of the same figure being referred to in different and very detached propositions. If however upon the whole, in this new appeal to the public judgment, the author should be found guilty of temerity, he will submit with patience; yet at the same time presumes to think, that upon this occasion, with more than usual reason, he hopes for candour.

# ELEMENTS

OF

# CYCLOMETRY.

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## DEFINITIONS.

(1.) The Sector of a Circle is an angular Figure, bounded by two Radii and an Arc of the Circle.

(2.) If the Arc equal one fourth part of the Circumference, the Sector is called a Quadrant, as A.B.C.A. Fig. 1. this will be most useful for our present purpose.

(3.) An Arcoid is a Sector with its Arc inverted, as in Fig. 2, A.*d*.C.B.A.

(4.) If in a Sector a Chord be drawn, the Area, contained between that Chord and the Arc, is a Segment.

Note 1. The *Sector* is used in measuring the apparent distance of two bodies, when the place of neither of them coincides with that of the observer; and is much employed at sea, in ascertaining the Longitude.

In taking the Altitude of objects, the *Arcoid* is resorted to; and is particularly useful, in Navigation, to ascertain the Latitude and apparent Time by the celestial bodies.

The *Segment* would be useful for various purposes, if the proportion between the Arc and the Chord, which subtends it, were accurately ascertained, from Geometrical Principles.

## PROPOSITION I.

The Area of a Quadrant, together with its own Arcoid, is equal to the Square of Radius.

Construction. Let A.C.B.D.A. Fig. 3, be the Square of Radius, then, from the opposite angles C. and D. describe the Arcs A.*e*.B. and A.*d*.B. within the Square.

Demonstration. Take away the Arcoid B.D.A.*e*.B. and there remains the Quadrant

A.C.B.*e*.A. Or take away the Arcoid A.C.B.*d*.A. the remainder is the Quadrant B.D.A.*d*.B.—Whence the Proposition is manifest.

Corollary I. The two opposite Arcoids are equal to each other.

Cor. II. The Quadrant is composed of two Segments under the Chord and Arc of 90° and one Arcoid.

## PROP. II.

The Arcoids about the Centre of a Circle are also equal.

Dem. About a Circle A.B.C.D. draw a Square of the Diameter, Fig 4. Bisect the sides; and, from those points, draw Diameters of the Circle. From each Angle of the Square E.F.I.G. with a Radius equal to that of the Circle, describe a Quadrant, so as to cut the Circle.—Then the Arcoids about the Centre having vertical angles, at O, and equal sides, are equal among themselves.

## PROP. III.

The Arcoid is equal to one third part of a Quadrant, and bisects the Isosceles Triangle,

formed by two Radii of the Circle and the Chord of ninety degrees.

Con. Let the Quadrant A.B.D.E. Fig. 5, be drawn. Draw also the Chord B.E. and the Line B.*m*. from the extreme of one Radius to the middle of the other.

Dem. Then is the Triangle A.B.E. equal to the Figure E.D.B.*m*. Take away the part E.B.*m*. and there remains the Segment E.D.B., equal to the Triangle *m*.B.A. or (as in Fig. 3) to the Arcoid A.C.B.*d*.A. equal to one third part of the Quadrant or to half the Triangle A.B.E.

Note. If *this Proposition be true*, THREE very important Corollaries most obviously follow from it, viz.

## COROLLARY I.

The Segment of a Circle, formed by the Chord and the Arc of ninety degrees, is equal to one fourth part of the Square of Radius.

## COROLLARY II.

The greatest inscribed Square of a Circle, together with four Segments formed by the Sides of

the Square, with the Circumference of the Circle, is equal to three times the Square of the Radius.

## COROLLARY III.

The Circle is equal to three times the Square of its Radius.

NOTE 2. *Since* however the truth of the Proposition is doubted, no use will be made of it in the following part of this essay, the object of which is to represent the principal steps by which the conclusion contained in the Proposition was arrived at, in order that if error lurk in any part of the argument it may be detected and exposed. Nevertheless it should be noticed that the argument relates only to the Segment of ninety degrees to the quadrantal Triangle and to the Square of Radius.

NOTE 3. It may also be proper to state here that, for the sake of conciseness, the word quadrantal, in this Essay, is used to denote that which is peculiar to the Quadrant of a Circle, as v.g. Quadrantal Triangle, Quadrantal Chord, Quadrantal Segment, &c. is the Triangle, Chord, Segment, &c. which is peculiar to the Arc of a Circle, con-

taining 90°, or one fourth part of the whole circumference.

## PROP. IV.

A Triangle, formed by the Diameter of a Circle and two equal Chords of the same Circle, is equal to the Square of Radius.

CON. Describe the Circle A.B.D.E. (Fig. 6.) Draw the Diameters A.D. and B.E. at right angles to each other. Draw also the two Chords of ninety degrees, B.A. and B.D. so as to form a Triangle in one of the Semicircles A.B.D.A. and about the other Semicircle A.E.D.A. construct a Paralellogram A.D.G.F.A.

DEM. Then is the Paralellogram evidently equal to the two Squares of Radius A.E. and E.D. and the Triangle A.B.D. standing on the same Base, and having the same altitude, is (by El. B. II. Th. 2. Cor. 2.) equal to half the Paralellogram or to the Square of Radius.

COR. I. The Square of Radius together with two Segments on the Chord of 90°, is equal to the Semicircle.

Cor. II. Twice the Square of Radius, less two Arcoids, is equal to the Semicircle.

## PROP. V.

The Chord of 90° in any Circle is equal to the Diagonal of its Square of Radius.

Construct as in Prop. IV. Fig. 6; and in one of the Squares A.E. draw the line A.E.; which is clearly both the Chord of 90° in the Circle A.B.D.E.A. and a Diagonal of the Square of its Radius A.*o*.

Definition 5. If, from any Point in the Circumference of a Circle, an Arc be described, so as to cut that Circumference in two other Points, each ninety degrees from the first, there will be formed a Crescent, called the *Hippocratic Lune*, and which in this Essay is always intended when the term Lune is used alone, or without any epithet.

Def. 6. The Segment formed by the Diameter of the Circle and the concave Arc of the Lune, is here called the *Lunar Segment*, to distinguish it from all others. When other Segments are referred to they are described according to their peculiar nature.

DEF. 7. The LUNETTE is a space bounded by the Quadrantal Arc of one Circle, and the Semicircumference of another Circle, described from the middle point of the Quadrantal Chord, with a Radius equal to half that Chord, and in such a manner as to include within itself the Quadrantal Arc first mentioned.

## PROP. VI.

The LUNE is equal to the Square of Radius.

CON. Describe a Circle A.B.C.D. Fig. 7. Draw the Diameter A.B. In one of the Semicircles, form an Isosceles Triangle A.B.C. having its vertex C. in the circumference: From the vertex, with a radius equal to one of the equal sides of the triangle, describe an Arc, cutting the circumference of the Circle in the points B. and A. Then will the two Arcs, which are on the same side of the Diameter, form a Lune L. whose area is equal to the Square of Radius.

DEM. Since (by El. B. VIII. The. 3.) all circles are in proportion to one another as the Squares of

their Radii, and since the Square of the Radius A.C. is equal to twice the Square of the Radius A.*o*. the Circles of which these are respectively Radii, are to each other in the same proportion, and therefore the Quadrant A.C.B *x*.A. is equal to either of the Semicircles A.*o*.B.C.A or A.*o*.B.D.A. Then from the Semicircle A.*o*.B.D.A and also from the Quadrant A.C.B.*x*.A. take away the Lunar Segment A.*o*.B.*x*.A., and there remains, in one case, the Lune A.D.B.*x*.A. and in the other the Triangle A.*o*.B.C.A. which (by Prop. IV.) is equal to the Square of the Radius A.*o*.

Cor. I. Hence the Lune, together with two Segments of 90° of the circle, to which it belongs, is equal to half that Circle.

Cor. II. Half the Lune is equal to the Quadrantal Triangle C.B.*o*.C. (Fig. 7.) of that Circle to which the Lune belongs, or of which it forms a part.

## PROP. VII.

The Lunette formed on a given Quadrant, is equal to half the Lune of the Circle to which that Quadrant belongs.

CON. Describe the Circle A.B.C.D. (Fig. 8.) of which the given Quadrant B.*o*.C.*n*.B. is a part. Extend the Radii, or sides of the Quadrant, B.*o*. and C.*o*. into Diameters of the Circle. Draw the Chord B.C. of the Quadrant; and with it describe an Arc, which shall form the Lune A.D.C.*x*.A. of the Circle. Also, from the middle point *m*. of the Chord, B.C. and with a Radius, equal to half that Chord, describe a Semicircle B.F.C.B. so as to include the Arc of the given Quadrant Then is the Lunette B.F.C.*n*.B. equal to half the Lune of the Circle A.B.C.D.

NOTE 3. If the different sections be marked *a.b.c.d.e.* they will be more readily distinguished from each other.

DEM. By the SIXTH Proposition, the Lunette *a* (Fig. 8.) is equal to the Triangle marked *c*. which also (by the same Proposition) is equal to the Semi-lune marked *e*. Whence the Proposition in hand is manifest.

NOTE 4. Hence it appears that the term Lunette is properly applied to a Geometrical Figure, only when that figure is compared with a

similar one of twice its own dimensions, when compared with a similar figure of half its own size, it may be called a Lune.

## PROP. VIII.

The Lunar Segment of a Circle is equal to twice a Segment on the Chord of 90° of that Circle.

Construct as in Prop. VI. Fig. 7, and from the Vertex C. of the Triangle draw the Radius C.*x*. so as to bisect both the Diameter A.B. and the Lunar Segment A.B.*x*.A.

DEM. By Prop. VI. Cor. I. the Semicircle A.B.D.A. is equal to the Lune L. together with two Segments of 90° of the Circle to which it belongs. Also the Semicircle is equal to the Lune L. together with the Lunar Segment A.B.*x*.A. therefore the Lunar Segment is equal to twice the Segment of the Circle on the Chord of 90°, i.e. either of the Segments A.C. or CB. is equal to either of the Semi-lunar Segments A.*x*.*o*.A. or *x*.*o*.B.*x*.

## PROB. A.

**To express in a Geometrical Form, different from the natural one, the value of a Quadrantal Segment.**

CON. As in Fig. 7, Take the Chord A.C. of the Quadrantal Segment given and with it as a Radius describe an Arc A.*x*. of forty-five degrees of a Circle. Draw the Sine A.*o*. and Versed-sine *o*.*x*. of the Arc *and the thing is done.*

DEM. By Prop. VIII. the Segment A.B.*x*.A. under the Chord and the Arc of 90°, is equal to twice the Segment A.C. or B.C. and therefore the Semi-segment, bounded by the Arc A.*x*. the Sine A.*o*. and the Versed-sine, *o*.*x*. is equal to either of the Segments A.C. or B.C.

NOTE 5. Here it might be easily shown that the value of any other Segment of a Circle, less than that comprehended under the Chord and Arc of 90°, may be represented in a similar manner by a corresponding Arc of the Lunar Segment and the Sine and Versed-sine of that Arc for

Construct as in Fig. 7, and from the angle of the Lune draw the Chords A.*m*. and A.D. (Fig. 9,) draw also the right line C.*m*. and perpendicular

to it A $r$. then as the Arc A.$m$. is to the Arc A.D. so are the corresponding Arcs of the Lunar Segment A.$n$. and A.$x$. to each other. So also, respectively, are the Segments of the Circle A.$m$. and A.D. to each other, and the similar portions of the Lunar Segment A.$r$.$n$. and A.$o$.$x$.

## PROB. B.

To express, Geometrically, the value of an Arcoid, in a form different from its natural one.

Con. Describe a Circle, (Fig. 10,) A.B.C.D.A. Draw two Diameters at right angles to each other, and from the extreme of one Diameter, with a Radius equal to the Chord of 90°, describe, in one of the Quadrants, an Arc, which shall form half the Lunar Segment $a$., and, in the same Quadrant, draw the Chord of 90° A.D. and *the thing is done.*

For by Prop. I. Cor. II. the Quadrant A.$o$.D.A. is composed of two Segments and an Arcoid, whence it follows (Prob. A.) that since the Parts marked $a$. and S. are (by Prop. VII.) each equal to a Segment, the part marked $e$. must be equal to an Arcoid of the same Circle.

NOTE 6. To shew that, in this Problem, there is no misapplication of terms, complete a Square D.F.C.O. on the Radius O.C. equal to A.O. and let the Square be divided into portions as in Prop. I. Fig. 3, mark the corresponding parts in the Quadrant O.D.C.O. with the same letters by which they are distinguished in the Quadrant A.O.D.A. then is it evident that the Arcoid *e.* in one Quadrant of the same Circle, is equal to the Arcoid *e.* in the other Quadrant, and that they differ from each other in form.

## PROP. IX.

The Quadrantal Segment is equal to the Arcoid.

CON. Describe a Circle A.B.C.D.A. (Fig. 11.) Divide it into four Quadrants, and construct about it a Square F.G.H.I. whose sides shall be Tangents to the Circle at A.B.C. and D. Then from two opposite Angles of the Square G.I. with a Radius equal to the Chord of 90°, describe, from the centre of the Circle, in one of its Quadrants, an Arc, which shall cut the circumference of the circle and terminate in one side of the Square.—

Also mark each section of the figure with distinguishing letters, *a.b.c.d.e.o.*

DEM. Since the mixtlinar right-angular spaces, *a.* and *o.* together; and *b.* and *e.* together are, by construction, half the Lunar Segment of the Circle A.B.C.D.A. each of these sums is equal (by Prob. A.) to a Segment of 90° of that Circle. But the three spaces marked *d.e.o.* together form the Arcoid of the quadrant, and the three spaces *a.c.b.* form the Quadrant itself; which by Prop. I. Cor. II. must be composed of two Segments and one Arcoid. The spaces *a.* and *b.* separately are each less than a Segment, and therefore the quantity *c.* must exceed the Arcoid by an equal value; whence it follows that the difference between *c.* and *d.* is equal to the sum of twice *o.* and twice *e.* together. Add the half of this sum, viz. *o.* and *e.* to the section *d.* to complete the Arcoid; also add one of the remaining quantities *o.* to the part *a.* and the other remaining quantity *e.* to the part *b.* and then each of the two Segments is complete. Now since when equal quantities are produced by the addition of equals, the original sums, to which they were respectively added, must have been

equal to each other, it follows that the part *a.* and the part *b.* must have been respectively equal to the part *d.* and that the Arcoid *d.o.e.* must be equal to the Segment *a.* + *o.* or *b.* + *e.*

NOTE 7. HENCE it appears that if with a Diagram, Fig. 12, (constructed as Fig. 11.) we describe an Arc A.*m.*D. from the angle H. as a centre, with a Radius equal to that of the circle, we shall cut off from the part *c.* a curviliniar triangle *m.* equal to that quantity by which the section *c.* exceeds the Arcoid *d.o.e.* and shall also obtain two quantities, *r.* and *s.* each equal to half the Quadrantal Segment of the Circle.

FOR the Figure H.A.*m.*D.H. is a Quadrant of the Circle; which, in this instance, is composed of two Arcoids, (viz. *d.o.e.* and *c.*) together with two other quantities, *r.* and *s.* each of them equal to each other, and to half a Segment, under the Chord and Arc of 90° of the Circle.

∴ Hence also it appears, since the Quadrant may consist indifferently either, of *one* Segment and two Arcoids, or of *one* Arcoid and two Segments, that the Arcoid is equal to the Segment; and that it is also equal to one third part of the Quadrant, conformable to Prop. III.

## PROP. X.

The Area of a Circle, whose Radius equals the Chord of 90° of another Circle, is equal to *twice* the Area of that other Circle.

Construct as in Prop. VI, Fig. 7.

DEM. Since in one of the two cases supposed, viz. when the Radius A.C. equals the Chord of 90°, the Square of Radius, or its equal A.C.B., with the Lunar Segment A,B.$x$.A. forms a Quadrant; and in the other case, the Square of Radius or its equal A.C.B. with two Segments A.C. and B.C. forms a Semicircle, it is evident that the wholes, of which these are respectively parts, are, to each other, in the ratio of *two* to *one*.

COR. I. Hence a Sphere whose Radius is equal to the Chord of 90° belonging to another Sphere, is equal to twice that other Sphere.

COR. II. Hence also a Cube, whose side is equal to the Chord of 90° of a circle, is double of another Cube, whose side is equal to the Radius of the same Circle.

## PROP. XI.

If the Radius of one Circle be equal to the Diameter of another, then is the Area of the former Circle equal to *four* times that of the latter.

Con. Describe an Isosceles right-angled Triangle A.B.C. (Fig. 13.) From the vertical angle B. as a centre, describe an Arc, passing through the angular points C. and A. Also, from the middle of the Base *n*. as a centre, describe an Arc, passing through the points B. and C. Moreover from the Middle *m*. of the Chord B.C. and with a Radius equal to half the Chord, describe upon it (externally of the Triangle) the Semicircle B.D.C. and draw its Chord B.D. of ninety degrees. Then is the Segment B.E.D.B. of the Semicircle B.D.C.B. equal to one fourth part of the Segment A.C.G.A. belonging to the Quadrant B.A.G.C.B.

Dem. The lines B.A. and B.C. being Radii of the same Circle, and at right angles to each other, the Figure B.A.G.C.B. is, by construction, a Quadrant; and A.C.G.A. its Segment. Also the Curve B.D.C. being a Semicircle, the line B.C. is its Diameter, and the Area B.E.D.B. forms a

Segment of 90° of that Semicircle. Moreover, since the Segment B.D.E.B. (or No. 1) is equal to half the Segment No. 2, (by Cor. II. to Prop. VI.) and this again is equal to half the Segment No. 3, it is evident that the Segment No. 1 is equal to one-fourth part of the Segment No. 3.— But since these Segments are similar parts of their wholes, the Circles also of which they are Segments must be in the same ratio.

Cor. From this and the preceding Proposition it appears that Circles, whose Radii are respectively equal to the Radius, the Quadrantal Chord, and the Diameter, of a given Circle, are to each other as the numbers *one*, *two* and *four*.

## PROB. C.

To divide half the Quadrantal Segment into two equal and dissimilar parts.

Construct as in Prop. XI. Fig. 13. Draw the Versed-sine *n*.G. dividing the Segment A.C.G.A. into two equal parts. Also from the middle *m*. of the Chord B.C. with a Radius equal to half that

Chord, describe the Arc C.$x$.$n$. within the Segment, then is the Semi-segment C.G.$n$.C. divided into two equal parts.

Dem. By Construction the Segment C.$n$.$x$.C. is equal to the Segment B.D.E.B. and (by Proposition XI.) this latter is equal to one fourth part of the Segment A.C.G.A. whose half therefore C.$n$.G.C. is evidently divided into two equal and dissimilar parts, of which the part C.$n$.$x$.C. is one, and C.G.$n$.$x$.C. the other.

## PROB. D.

To divide an Arcoid into two equal parts.

Construct as in Prop. VII. Fig. 8. Draw the Chord B.A. Form the Lunette A.E.B.A. Complete the Square of Radius A.$o$.B.E.A. and draw the Diagonal E.O. Divide half the Lunar Segment A.$o$.$x$.A. (by Prob. C.) into two equal and dissimilar parts. On the Chord D.A. from the point D. towards A. take D.$s$. equal to A.E. and on the Radius D.$o$. from D. take D.$z$. equal to E.$r$. Make $z$.$t$. equal to the Radius D.$o$. and, with that Radius, from the point $t$. describe

the Arc *z.s.* which, it is evident, will pass through both the points *z.* and *s.* and form a Figure *s.*D.*z.s.* in all respects similar and equal to the Semi-Arcoid, A.E.*r*.A. In the spaces *i.n.p.m.* and *q.* place distinguishing letters then

DEM. By Construction, we have the Triangle A.E.*o*.A. equal the Triangle A.*o*.D.A. the parts *m.* and *i.* equal to each other; and we also have (by Prob. C.) the part *n.* equal to the part *p.* and therefore the part *q.* or its equal, the sum of *n.* and *i.* equal to the mixtliniar Figure A.*s.z.o.*A. equal to the Square of Radius divided by four.

Moreover the Quadrantal Segment *p.* or its representative *n.* is equal (by Prob. A.) to the Figure A.*x.o.* whence it is evident that the Semi-arcoid *i.* equal to *m.* is also equal to the Figure A.*s.z.x.*A. and hence it appear that the Arcoid of a Semi-lune A.*s.*D.*x.*A. is divided into two equal parts *s.*D.*z.s* and A.*s.z.x.*A. which was to be done.

NOTE 7. From this Problem it follows that the Segment of a Circle on the Chord A.B. of 90° is equal to the Arcoid A.*r*.B.E.A. that each of them is equal to one-third part of the Quadrant A.*o*.B.A.

and that the doctrine maintained in the *Third* Proposition and its Corollaries is true. It may be useful however to pursue these Equations a little further.

## PROP. XII.

The Lunette is equal to the Lunar Segment of a Circle.

DEM. For in Fig. 8, the Segment on the Chord A.E. and the Semi-arcoid *i.* (evidently composing half the Lunette) are together equal (Prob. D.) to the portions *p.* and A.*x*.O.A. which evidently compose half the Lunar Segment A.*x*.C.A.— Their wholes, therefore, the Lunette and the Lunar Segment, must also be equal.

COR. Hence the Segment is equal to half the Lunette which incloses it.

## PROP. XIII.

The Lunette on the Chord of 90° is equal to half the Square of Radius.

DEM. The Lunette (by Prop. VII.) is equal to half the Lune, the whole of which (by Prop. VI.) is equal to the Square of Radius or twice the Lunette.

COR. Hence the Lunar Segment is equal (by Prop. XII.) to half the Square of Radius.

NOTE 8. By the preceding Propositions we are enabled to express the value of the Square of Radius, and the proportional parts of it, in a great variety of useful forms.

## PROB. E.

To divide a Circle into three parts, each of them equal to the Square of Radius.

CON. Describe a Circle A.B.C.D.A. Fig. 14. Divide the circumference into four equal parts at the points A.B.C.D. and from any two opposite points B.D. with a Radius equal to a Chord of 90°, describe an Arc cutting the circumference in the two other points A. and C. *and the thing is done.*

DEM. By Def. V. and Prop. VI. each of the Lunes *a.* and *b.* is equal to the square of Radius;

and the intermediate space *c.* is equal to two Lunar Segments, or to four Quadrantal Segments, which (by Cor. to Prop. XIII.) are, together, equal to the Square of Radius, or to each of the Lunes *a* and *b.* respectively.

## PROP. XIV.

The Cusp of a Lune cut off by a right Line drawn from the Centre of the concave Arc of the Lune, is equal to a Triangle formed by the Chord of the Cusp, by a perpendicular let fall from the point of the Cusp upon the Line itself, and by the part of that Line intercepted between the perpendicular and the Chord.

Construct as in Prob. A. Fig 9.

DEM. Then since by Prob. A. the Segment A.*m.* of the Cusp is equal to the mixtliniar Figure A.*r.n.*A. take away the Segment A.*m.* from the Cusp, and the Figure A.*r.n.*A. from the Triangle, and there remains A.*n.m.*A. common to them both. Whence it follows that the Cusp and the Triangle are equal to each other.

NOTE 9. Hence if the Chord A.C. be drawn, and upon it a Triangle A.*s*.C. be constructed (El. B. VI.) equal to the Triangle A.*r*.*m*. or to its equal the Cusp A.*n*.*m*. then will the remaining part *m*.*n*.*x*.D. of the Semi-lune be equal (by Prop. VI. Cor. II.) to the remaining part *s*.*o*.C. of the Quadrantal Triangle A.*o*.C.

Moreover it is well known to Geometricians that a perpendicular let fall from the Vertex *m* upon the Radius A.*o*. will touch the point *s*. whence the right Line *s*.C. must be drawn to form a Triangle A.*s*.C. equal to the given Triangle A.*r*.*m*.A.

Hence also it appears that since (by Cor. to Prop. XIII.) the Semi-lune A.*x*.D.*m*.A. is double of the Semi-lunar Segment A.*o*.*x*.*n*.A. the Cusp of the Lune A.*n*.*m*.A. cut off by the right Line C.*m*. is double of that part of the lunar Segment A.*e*.*n*.A. cut off by the same right Line. And moreover that the remaining part *m*.*n*.*x*.D.*m*. of the Lune is double of the remaining part *n*.*e*.*o*.*x*.*n*. of the Lunar Segment.

## PROP. XV.

A Cusp, cut off from the Lune of a Circle by a right Line, drawn from the Centre of the concave Arc, is equal to a Lune formed upon the Chord of the Cusp.

CON. Describe a Circle A.B.C.D. (Fig. 15.) and form the Lune of it. Draw any Chord D.B. from the Centre D. of the concave Arc A.*n*.C. so as to cut the Lune L. and form a Cusp A.*n*.B.A. Draw also the Chord A.B. of the Cusp, and upon that Chord form a Lunette according to Definition VII.

Also from the point A. of the Cusp draw A.I. perpendicular to D.B. and from I. as a Centre with the Radius A.I. form the Lune *t*. then is the Cusp equal to that Lune.

DEM. It might very easily be proved that the Line A.I. drawn at right angles to B.I. must also be equal to B.I. and that the circumference of a Circle drawn from the middle of the base A.B. would pass through the points A.B. and I. and therefore (by Prop. VI.) the Lune *t*. is equal to

the Triangle A.I.B. which also (by Prop. XIV.) is equal to the Cusp; wherefore the Cusp A.n.B.A. is equal to the Lune t. formed upon the Chord A.B.

POSTULATE. About any Triangle (by El. B. V. Prop. 23) a Circle may be described, whose circumference shall pass through the angular points of the Triangle.

## PROP. XVI.

The Lunettes formed upon the two equal sides of a right angled Triangle are together equal to the Area of that Triangle.

NOTE. This Proposition includes two cases.

### CASE I.

When the Triangle is Isosceles.

Construct as in Prop. VII. Fig. 8.

DEM. By Prop. VII. the Lunette *a.* is equal to the Semi-lune *e.* and therefore by Prop. VI. equal to the Triangle *c.* which by construction is

equal to half a right angled Isosceles Triangle included in a Semicircle A.B.C. and therefore the two Lunettes upon the two equal sides of such a Triangle are together equal to the whole.

## Case II.

When the Triangle has unequal sides.

Con. Describe a Circle A.B.C.D.A. (Fig. 16) draw the Diameter A.C. and upon it construct the right angled Triangle A.B.C. Bisect the sides A.B. and B.C. containing the right angle, and describe upon each a Semicircle A.*e*.B. and B.*f*.C. then is the sum of the Lunettes *z*. and *w*. equal to the Triangle T.

Dem. The two Semicircles A.*e*.B.A. and B.*f*.C.B. are together equal to the Semicircle A.C.D.A. or the Semicircle A.B.C.A. From these equal quantities take away the two Segments *m*. and *g*. and there remains, in one case, the two Lunettes *z*. and *w*. and in the other case, the Triangle T. which is therefore equal to the sum of the two Lunettes.

## PROP. XVII.

Twice the Area of a Segment, bounded by the side of an Octagon, and an Arc of the circumscribing Circle is equal to the space contained between half the Arc of the Lunar Segment and a Line drawn from the angle of that Segment to the middle point of the Arc.

CON. Describe the Circle A.E.C.D. (Fig. 17,) Draw the Diameters A.C. and D.E. at right angles to each other, and form the Lunar Segment A.C.*x*.A. Bisect the Arc A.E. in B. and draw the Chords A.B. and B.E. Draw also the Chord of the Circle D.B. and the Chord A.*x*. of the Lunar Segment, then are the two Segments *e*. and *e*. of the Circle equal together to the space A.*x*.*i*.A. of the Lunar Segment.

DEM By Prob. A. each of the Segments marked *e*. is equal to either of the spaces marked *o*. and *u*. whence the Proposition is manifest.

COR. I. If the lines B.*x*.C. and *x*.*n*. be drawn we have three triangles, all in the same Quadrant, equal to each other respectively; viz. A.B.*n*. *n*.B.*x*.

and $x$.B.E. also each of the sides A.B., B.$n$., B $x$. and B.E. are equal each to each, and the sides A.$n$., $n$.$x$., and $x$.E. are equal to each other, being also sides of an Octagon whose Centre is B.— From the point C. with the Radius C.E. describe the Arc E.$p$. Then from each of the Triangles A.B.$n$. $n$.B.$x$. and $x$.B.E. take away the part marked $c$. and there remain three Triangles equal among themselves, and each of them equal also (by Prop. XIV.) to the Cusp A.$c$.B.A. of the Lune cut off by the Chord D.B.

Moreover draw the Chord E.C. and two sides of an Octagon E.K. and K.C. then we have the Triangle $f$. equal to the Triangle $g$. or the Triangle A.H.$x$.A. included within the Semi-lunar Segment.

Cor. II. Hence we may express, in an integral part of the Square of Radius, the value of the two Segments $e$. and $e$. for bisect the Radius A.H. in $q$. and join D.$q$. then is the Triangle D.$q$.H. equal (Prop. XIII.) to the Semi-lunar Segment A.H.$x$.A. equal by construction to the Square of Radius divided by four, and since H.$n$. is equal to H.$x$. we have the Triangle H.$n$ D.

equal to the Triangle H.*r*.A. whence it is plain that the Triangle *q*.*n*.D. must express the value of the two parts of the Lunar Segment *u*. and *o*. or the two Segments of the circle *e*. and *e*. together.

Note 10. By the same means may be expressed, in terms of the Square of Radius, the value of any number of unequal Segments contained in the Arc of a Semicircle.

## PROP. XVIII.

A right angled Triangle is less than the Square of half its base by a Rectangle under the Radius of the Circle, which circumscribes the Triangle, the difference between that Radius and the Altitude of the Triangle.

Con. Describe a Circle A.B.C.D. (Fig. 18) and divide it into Quadrants. From one extreme of a Diameter A.C. draw any Chord C.B. and join B.A. then is A.B.C. a Triangle in a Semicircle, described from the middle of the Base A.C. and therefore the Angle A.B.C. is a right-angle. Upon the Radius A.O. complete the Square A.O.E.F. From B. the Vertex of the Triangle let fall a perpendicular B.*s*. upon the side of the

Square A.O. also through B. and parallel to A.O. draw the line G.H. then is the Rectangle G.O. equal to the Triangle A.B.C. and less than the Square of half the Base A.O. by a Rectangle under the Line G.H. equal to A.O. and G.F. equal the difference between the Radius A.O. or it equal A.F. and A.G. or B.s. the Altitude of the Triangle.

DEM. Join A.E. and E.C. also A.H. and H.C. then the Square A.F.E.O. standing on half the Base A.C. of the Triangle A.E.C. and having the same Altitude, is equal (by El.) to that Triangle, and, for the same reason, the Rectangle G.O. is equal to either of the Triangles A.H.C. or A.B.C. and therefore the difference between the Squares of the Radius A.O. and the right angled Triangle A.B.C. is equal to F.H. a Rectangle (as already stated) under the Radius, and the difference between the Radius and the Altitude of the Triangle, according to the terms of the Proposition.

COR. I. Hence it is evident that the two Triangles A.H.E. and C.H.E. formed by the Co versed Sine of the Angle made by the Base

A.E. and the Hypotheneuse C.B. of the Triangle, the Chord of 90° of the circumscribing Circle, and a line joining their extremes), are together equal to a Rectangle under the Radius and the co-versed Sine.

Cor. II. Hence also since the Lunettes formed on the two sides, which contain the right angle, are together equal to the triangle, it follows that they are together less than the Square of Radius, or the Lune of the Circle which circumscribes the triangle, by a rectangle under the Radius of the Circle and the Co-versed Sine of the Arc whose Chord is a diameter to the Semicircle of the smaller Lunette. Or on the sides of the triangle A.B.C. containing the right angle, and also on the Chord A.E form Lunettes, then since the triangle A.E.O.A. is equal to the Lunette A.F.E.A. and half the sum of the Lunettes *m.* and *n.* is equal to half the triangle A.B.C. or half the rectangle F.H. it follows that the difference between half the sum of the Lunettes *m.* and *n.* and the Lunette A.F.E.A. is equal to the triangle A.E.H. or half the rectangle F.H. (bounded by the Chords E.C., C.B. and the Arc B.E.

Cor. III. Hence also since the mixtliniar space E.C.B. and the rectangle F.H. both vary as the difference between the Radius and the Altitude of the circumscribed triangle, it follows that the space E.C.B. is equal to the two triangles together A.E.H. and C.E.H.

## PROP. XIX.

The Cusp whose longest Chord is the side of an Hexagon, is equal to the Square of the Chord divided by four.

Con. Describe the circle A.B.C.D. (Fig. 19.) draw two diameters A.C. and D.B. at right angles to each other: draw also, in one of the Semicircles, the Chords A.E., E.F. and F.C. all equal to each other. From the point D. draw the lines D.E. and D.F. and describe the Arc of the Lunar Segment A.*m*.*n*.C. and, perpendicular to D.E. draw the line A.*r*′. Moreover, upon the Chord E.F. construct a Square E.G.H.F. and draw the two diagonals F.G. and E.H. bisecting each other in *x*. Then because E.F. is one side of half an Hexagon it is equal (El. B. V. Th. 29. Cor.)

to the Radius A.O. whence (by Prop. VI.) the Square E.H. is equal to the Lune of the Circle A.B.C.D.

Dem. By construction the Chord A.E. is the longest which can be drawn in the Cusp A.*m*.E.A. and equal to E.F. Also it may be easily proved, that, the lines A.*r*. and *r*.E. are respectively equal to the lines G.*x*. and *x*.E. and that E.G is equal to A.E. whence the triangles E.G.*x*.E. and E.A.*r*.E. are equal and similar in all respects.—But the triangle G.*x*.E. is equal the Square of A.E. divided by four, and the triangle A.*r*.E. is equal (Prop. XIV.) to the Cusp A.*m*.E.A. whence the Cusp is equal to the Square of its own longest Chord divided by four.

Cor. Hence it appears that if the Chord of the Cusp equal the Radius of the convex Arc, the Cusp is equal to one fourth part of the Lune to which it belongs.

## CONCLUSION.

In a manner similar to this, the side of various Polygons may be applied to the circumference of a Circle, and, if those from the Decagon to the

Trigon be made the Chord of different Cusps, they will exhibit some singular relations both to the Circle and the Square of Radius; they show that the Lune is applicable to every kind of Triangle, and lead to some useful investigations in the higher branches of Practical Mathematics. But a field so wide and so rich as this appears to be, if not already cultivated, calls for and will undoubtedly secure the attention of more active and scientific labourers.

THE END.

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